Prime counting function $\pi(x)$

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Abstract

Natural numbers are created by humans in imitation of nature.

Therefore, the world of natural numbers is the world of nature.

In the natural world, natural laws are discovered, not proven.

In this case, the prime number theorem is a prime number law, not proved, but discovered.

"$\frac{1}{\ln p}$ is the probability that a natural number p is prime" is the law of natural numbers (prime number theorem).

The above law of natural numbers (prime number theorem) is found as follows.

The prime number counting function $\pi(x)$ is the integral of the above probability $\frac{1}{\ln p}$.

The above integral is a series of finite terms.

The prime number counts calculated from the above series are almost identical to the prime numbers on Wikipedia.

Thus, the law of the above natural numbers (prime number theorem) is discovered.

1. Beginning

Natural numbers are created by humans in imitation of nature.

Therefore, the world of natural numbers is the world of nature.

In the natural world, natural laws are discovered, not proven.

In this case, the prime number theorem is a prime number law, not proved, but discovered.

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The above law of natural numbers (prime number theorem) is found as follows.

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Thus, the law of the above natural numbers (prime number theorem) is discovered.

2. Discovery of the probability $\frac{1}{\ln p}$

There are always natural numbers m and n that hold the following inequality for natural number p greater than or equal to natural number 2.

e is Napier's number.

$$e^m < p^n < e^{m+1}$$

When converted to logarithms, the following inequality holds.

$$m < n \ln p < (m+1) \qquad \ln e = 1$$

$$1 < \frac{n \ln p}{m} < (1 + \frac{1}{m})$$

When natural number m is infinite, the following equation holds.

$$\lim_{m \to \infty} \frac{n \ln p}{m} = 1$$

$$\lim_{m \to \infty} \frac{n}{m} = \frac{1}{\ln p}$$

There are a number $\pi(x)$ of prime numbers $p_1, p_2, p_3 \dots p_i \dots p_\pi$ from natural number 2 to natural number x.

For the above prime number p_i, the following equation holds.

$$\frac{1}{\ln p_i}\ln p_i = 1$$

$$\sum_{i=1}^{\pi}\frac{1}{\ln p_i}\ln p_i = \pi$$

Then, the above numerical value 1 indicates the existence of a number 1 of prime, that is, the probability 1 of the existence of prime.

That is, $\frac{1}{\ln p}$ is the probability (proportion or ratio) that a natural number p is prime.

3. Derivation of prime counting function $\pi(x)$

The prime counting function $\pi(x)$ is the integral from the natural number 2 to the natural number x of the above probability $\frac{1}{\ln p}$.

$$\pi(x) = \sum_{p=2}^{x}\frac{1}{\ln p} = \int_2^x \frac{1}{\ln p}dp$$

The above integration of the probability $\frac{1}{\ln p}$ is the following series (2.1).

$$\frac{d}{dp}\left(\frac{p}{\ln p}\right) = \frac{1}{\ln p} - \frac{p}{(\ln p)^2}\frac{1}{p}$$

$$= \frac{1}{\ln p} - \frac{1}{(\ln p)^2}$$

$$\frac{1}{\ln p} = \frac{d}{dp}\left(\frac{p}{\ln p}\right) + \frac{1}{(\ln p)^2}$$

$$\pi(x) = \int_2^x \frac{1}{\ln p}dp = \int_2^x \frac{d}{dp}\left(\frac{p}{\ln p}\right)dp + \int_2^x \frac{1}{(\ln p)^2}dp$$

$$\pi(x) = \int_2^x \frac{1}{\ln p}dp = \frac{x}{\ln x} + \int_2^x \frac{1}{(\ln p)^2}dp$$

$$\int_2^x \frac{1}{(\ln p)^2}dp = \frac{x}{(\ln x)^2} + 2\int_2^x \frac{1}{(\ln p)^3}dp$$

$$\pi(x) = \int_2^x \frac{1}{\ln p}dp = \frac{x}{\ln x} + \frac{x}{(\ln x)^2} + 2\int_2^x \frac{1}{(\ln p)^3}dp$$

$$2\int_2^x \frac{1}{(\ln p)^3}dp = \frac{2x}{(\ln x)^3} + 6\int_2^x \frac{1}{(\ln p)^4}dp$$

$$\pi(x) = \int_2^x \frac{1}{\ln p}dp = \frac{x}{\ln x} + \frac{x}{(\ln x)^2} + \frac{2x}{(\ln x)^3} + 6\int_2^x \frac{1}{(\ln p)^4}dp$$

If the above is continued, the following series (2.1) is obtained.

$$\pi(x) = \int_2^x \frac{1}{\ln p}dp = \frac{x}{\ln x} + \sum_{n=1}^{m} \frac{n!x}{(\ln x)^{n+1}} \qquad (2.1)$$

The natural number m is calculated as follows:

$$\frac{d}{dx}\left(\frac{m!x}{(\ln x)^{m+1}}\right) = \frac{m!}{(\ln x)^{m+1}} - \frac{(m+1)!x}{(\ln x)^{m+2}}\frac{1}{x} = 0$$

$$1 - \frac{m+1}{\ln x} = 0 \qquad m = \ln x - 1$$

4. 5 examples of prime count calculated based on the above series (2.1) are shown below.

[1]

$x = 10^2 \qquad m = \ln x - 1 = \ln 10^2 - 1 = 3.$

Wikipedia $\pi(10^2) = 25$

$$\pi(x) = \frac{x}{\ln x} + \sum_{n=1}^{2} \frac{n!x}{(\ln x)^{n+1}} = 27$$

$$\frac{x}{\ln x} = 21.$$

$$\frac{x}{(\ln x)^2} = \frac{10^2}{(\ln 10^2)^2} = 4.$$

$$\frac{x}{(\ln x)^2} + \frac{2x}{(\ln x)^3} = \frac{10^2}{(\ln 10^2)^2} + \frac{2\times 10^2}{(\ln 10^2)^3} = 6.$$

[2]

$x = 10^3 \qquad m = \ln x - 1 = \ln 10^3 - 1 = 5.$

Wikipedia $\pi(10^3) = 168.$

$$\pi(x) = \frac{x}{\ln x} + \sum_{n=1}^{2} \frac{n!x}{(\ln x)^{n+1}} = 171.$$

$$\frac{x}{\ln x} = 144.$$

$$\frac{x}{(\ln x)^2} = \frac{10^3}{(\ln 10^3)^2} = 20.$$

$$\frac{x}{(\ln x)^2} + \frac{2x}{(\ln x)^3} = \frac{10^3}{(\ln 10^3)^2} + \frac{2\times 10^3}{(\ln 10^3)^3} = 27.$$

[3]

$x = 10^{10}$ $\quad m = \ln x - 1 = \ln 10^{10} - 1 = 22.$

Wikipedia $\quad \pi(10^{10}) = 455{,}052{,}511$

$$\pi(x) = \frac{x}{\ln x} + \sum_{n=1}^{7} \frac{n!x}{(\ln x)^{n+1}} = 455{,}055{,}222.$$

$$\frac{x}{\ln x} = 434{,}294{,}481.$$

$$\sum_{n=1}^{2} \frac{n!x}{(\ln x)^{n+1}} = \frac{10^{10}}{(\ln 10^{10})^2} + \frac{2\times 10^{10}}{(\ln 10^{10})^3} = 20{,}499{,}430.$$

$$\sum_{n=1}^{3} \frac{n!x}{(\ln x)^{n+1}} = \frac{10^{10}}{(\ln 10^{10})^2} + \frac{2\times 10^{10}}{(\ln 10^{10})^3} + \frac{6\times 10^{10}}{(\ln 10^{10})^4} = 20{,}712{,}876.$$

$$\sum_{n=1}^{4} \frac{n!x}{(\ln x)^{n+1}} = \frac{10^{10}}{(\ln 10^{10})^2} + \cdots + \frac{24\times 10^{10}}{(\ln 10^{10})^5} = 20{,}749{,}955.$$

$$\sum_{n=1}^{5} \frac{n!x}{(\ln x)^{n+1}} = \frac{10^{10}}{(\ln 10^{10})^2} + \cdots + \frac{120\times 10^{10}}{(\ln 10^{10})^6} = 20{,}758{,}006.$$

$$\sum_{n=1}^{6} \frac{n!x}{(\ln x)^{n+1}} = \frac{10^{10}}{(\ln 10^{10})^2} + \cdots + \frac{720\times 10^{10}}{(\ln 10^{10})^7} = 20{,}760{,}104.$$

$$\sum_{n=1}^{7} \frac{n!x}{(\ln x)^{n+1}} = \frac{10^{10}}{(\ln 10^{10})^2} + \cdots + \frac{5040\times 10^{10}}{(\ln 10^{10})^8} = 20{,}760{,}741.$$

[4]

$x = 10^{20}$ $\quad m = \ln x - 1 = \ln 10^{20} - 1 = 45.$

Wikipedia $\quad \pi(x = 10^{20}) = 2{,}220{,}819{,}602{,}560{,}918{,}840.$

$$\pi(x) = \frac{x}{\ln x} + \sum_{n=1}^{8} \frac{n!x}{(\ln x)^{n+1}} = 2{,}220{,}819{,}601{,}694{,}094{,}327.$$

$$\frac{x}{\ln x} = \frac{10^{20}}{\ln 10^{20}} = 2{,}171{,}472{,}409{,}516{,}259{,}138.$$

$$\sum_{n=1}^{1} \frac{n!x}{(\ln x)^{n+1}} = \frac{10^{20}}{(\ln 10^{20})^2} = 47{,}152{,}924{,}252{,}903{,}482.$$

$$\sum_{n=1}^{2} \frac{n!x}{(\ln x)^{n+1}} = \frac{10^{20}}{(\ln 10^{20})^2} + \frac{2\times 10^{20}}{(\ln 10^{20})^3} = 49{,}200{,}749{,}733{,}767{,}281.$$

$$\sum_{n=1}^{3}\frac{n!x}{(\ln x)^{n+1}}=\frac{10^{20}}{(\ln 10^{20})^2}+\cdots+\frac{6\times 10^{20}}{(\ln 10^{20})^4}=49{,}334{,}153{,}629{,}703{,}285.$$

$$\sum_{n=1}^{4}\frac{n!x}{(\ln x)^{n+1}}=\frac{10^{20}}{(\ln 10^{20})^2}+\cdots+\frac{24\times 10^{20}}{(\ln 10^{20})^5}=49{,}345{,}740{,}944{,}877{,}165.$$

$$\sum_{n=1}^{5}\frac{n!x}{(\ln x)^{n+1}}=\frac{10^{20}}{(\ln 10^{20})^2}+\cdots+\frac{120\times 10^{20}}{(\ln 10^{20})^6}=49{,}346{,}999{,}021{,}637{,}187.$$

$$\sum_{n=1}^{6}\frac{n!x}{(\ln x)^{n+1}}=\frac{10^{20}}{(\ln 10^{20})^2}+\cdots+\frac{720\times 10^{20}}{(\ln 10^{20})^7}=49{,}347{,}162{,}934{,}375{,}593.$$

$$\sum_{n=1}^{7}\frac{n!x}{(\ln x)^{n+1}}=\frac{10^{20}}{(\ln 10^{20})^2}+\cdots+\frac{5040\times 10^{20}}{(\ln 10^{20})^8}=49{,}347{,}187{,}849{,}614{,}824.$$

$$\sum_{n=1}^{8}\frac{n!x}{(\ln x)^{n+1}}=\frac{10^{20}}{(\ln 10^{20})^2}+\cdots+\frac{40320\times 10^{20}}{(\ln 10^{20})^9}=49{,}347{,}192{,}177{,}835{,}189.$$

［5］

$x=10^{28}\qquad m=\ln x-1=\ln 10^{28}-1=63.$

Wikipedia $\pi(10^{28})=157{,}589{,}269{,}275{,}973{,}410{,}412{,}739{,}598$

$$\pi(x)=\frac{x}{\ln x}+\sum_{n=1}^{6}\frac{n!x}{(\ln x)^{n+1}}=157{,}589{,}107{,}496{,}661{,}184{,}852{,}886{,}301.$$

$$\frac{x}{\ln x}=155{,}105{,}172{,}108{,}304{,}224{,}161{,}117{,}471.$$

$$\sum_{n=1}^{1}\frac{n!x}{(\ln x)^{n+1}}=\frac{x}{(\ln x)^2}=\frac{10^{28}}{(\ln 10^{28})^2}=2{,}405{,}761{,}441{,}474{,}667{,}464{,}540{,}316.$$

$$\sum_{n=1}^{2}\frac{n!x}{(\ln x)^{n+1}}=\frac{10^{28}}{(\ln 10^{28})^2}+\frac{2\times 10^{28}}{(\ln 10^{28})^3}=2{,}480{,}390{,}649{,}960{,}957{,}535{,}973{,}511.$$

$$\sum_{n=1}^{3}\frac{n!x}{(\ln x)^{n+1}}=\frac{10^{28}}{(\ln 10^{28})^2}\ldots+\frac{6\times 10^{28}}{(\ln 10^{28})^4}=2{,}483{,}863{,}262{,}828{,}929{,}297{,}882{,}432.$$

$$\sum_{n=1}^{4}\frac{n!x}{(\ln x)^{n+1}}=\frac{10^{28}}{(\ln 10^{28})^2}\ldots+\frac{24\times 10^{28}}{(\ln 10^{28})^5}=2{,}483{,}917{,}124{,}850{,}584{,}525{,}088{,}882.$$

$$\sum_{n=1}^{5}\frac{n!x}{(\ln x)^{n+1}}=\frac{10^{28}}{(\ln 10^{28})^2}\ldots+\frac{120\times 10^{28}}{(\ln 10^{28})^6}=2{,}483{,}933{,}833{,}406{,}862{,}395{,}538{,}927.$$

$$\sum_{n=1}^{6}\frac{n!x}{(\ln x)^{n+1}}=\frac{10^{28}}{(\ln 10^{28})^2}\ldots+\frac{720\times 10^{28}}{(\ln 10^{28})^7}=2{,}483{,}935{,}388{,}356{,}960{,}691{,}768{,}830.$$

The following example $\pi(10^{30})$ is a prediction that has not yet been published on Wikipedia.

$x = 10^{30}$ $\quad m = \ln x - 1 = \ln 10^{30} - 1 = 69.$

$$\pi(x) = \frac{x}{\ln x} + \sum_{n=1}^{6} \frac{n!x}{(\ln x)^{n+1}} = 14{,}692{,}398{,}886{,}701{,}008{,}787{,}306{,}610{,}970.$$

$$\frac{x}{\ln x} = \frac{10^{30}}{\ln 10^{30}} = 14{,}476{,}482{,}730{,}108{,}394{,}255{,}037{,}630{,}630.$$

$$\sum_{n=1}^{1} \frac{n!x}{(\ln x)^{n+1}} = \frac{x}{(\ln x)^2} = \frac{10^{30}}{(\ln 10^{30})^2} = 209{,}568{,}552{,}235{,}126{,}588{,}022{,}178{,}699.$$

$$\sum_{n=1}^{2} \frac{n!x}{(\ln x)^{n+1}} = \frac{10^{30}}{(\ln 10^{30})^2} + \frac{2\times 10^{30}}{(\ln 10^{30})^3} = 215{,}636{,}183{,}289{,}537{,}845{,}978{,}110{,}171.$$

$$\sum_{n=1}^{3} \frac{n!x}{(\ln x)^{n+1}} = \frac{10^{30}}{(\ln 10^{30})^2} \ldots + \frac{6\times 10^{30}}{(\ln 10^{30})^4} = 215{,}899{,}697{,}158{,}053{,}407{,}865{,}503{,}980.$$

$$\sum_{n=1}^{4} \frac{n!x}{(\ln x)^{n+1}} = \frac{10^{30}}{(\ln 10^{30})^2} \ldots + \frac{24\times 10^{30}}{(\ln 10^{30})^5} = 215{,}914{,}956{,}173{,}920{,}246{,}208{,}658{,}270.$$

$$\sum_{n=1}^{5} \frac{n!x}{(\ln x)^{n+1}} = \frac{10^{30}}{(\ln 10^{30})^2} \ldots + \frac{120\times 10^{30}}{(\ln 10^{30})^6} = 215{,}916{,}060{,}658{,}318{,}619{,}884{,}878{,}431.$$

$$\sum_{n=1}^{6} \frac{n!x}{(\ln x)^{n+1}} = \frac{10^{30}}{(\ln 10^{30})^2} \ldots + \frac{720\times 10^{30}}{(\ln 10^{30})^7} = 215{,}916{,}156{,}592{,}614{,}532{,}268{,}980{,}340.$$

5. Conclusion

As described above, the prime count of the prime counting function $\pi(x)$ can be calculated from the above series (2.1).

In addition, it was probabilistically proved that "a number 1 of prime number appear in natural numbers at approximately intervals $\ln p$", which is empirically known.

Since prime numbers occur probabilistically, it is impossible to predict their occurrence.

6. References

1. Erdős, Paul (1949-07-01), "On a new method in elementary number theory which leads to an elementary proof of the prime number theorem,", Proceedings of the National Academy of Sciences (U.S.A.: National Academy of Sciences) 35 (7): 374-384, doi:10.1073/pnas.35.7.374